Institute of Actuaries

Students' Society

(Interest and Life Contingencies)

(Notes and papers ...)

Institute of Actuaries Students' Society

INTEREST *and* LIFE CONTINGENCIES

NOTES AND PAPER ARRANGED BY THE INSTITUTE OF ACTUARIES STUDENTS' SOCIETY

LONDON:
SANDERS PHILLIPS & CO. LTD
6 & 8, Upper Thames Street, E.C.4
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PARTS I. AND II. OF THE 1918 SYLLABUS

I. INTRODUCTORY.

The reading recommended by the Board of Examiners is as follows :—

PART I.—SECTION A.

(1) H. S. Hall and S. R. Knight: "Higher Algebra." (Macmillan & Co., Ltd.)

No questions will be set on Interest and Annuities, Advanced Convergency and Divergency of Series, Continued Fractions, Indeterminate Equations of the Second Degree, Theory of Numbers, Inverse Probabilities, Determinants, Elimination, or Cubic and Biquadratic Equations.

(2) J. Burn and E. H. Brown: "Elements of Finite Differences," Part I. (C. & E. Layton.)

Institute *Text-Book*, Part II., chapters xxii., xxiii. and xxiv., sections 1-20.

J. Edwards: "Differential Calculus for Beginners." (Macmillan & Co., Ltd.)

Chapters i.—vii. *and* xiii, *omitting Trigonometrical references.*

J. Edwards: "Integral Calculus for Beginners." (Macmillan & Co., Ltd.)

Chapters i.—vi. *and the general propositions in Chapter* viii., *omitting Trigonometrical references.*

Institute *Text-Book*, Part I., chapter ix.

J.I.A.,* vol. xl., p. 116, section 1 (W. P. Elderton, *Approximate Summation*); vol. xliv., p. 402, section 1-5 (G. J. Lidstone and S. E. Macnaghten, *Integration by Parts*).

SECTION B.

Institute *Text-Book*, Part I. (excluding chapters ix. and x.).

PART II.

Institute *Text-Book*, Part II. (excluding chapters xix., xx,, xxii., xxiii. and xxiv., sections 1-20).

J.I.A., vol. xxii., p. 407 (T. B. Sprague, *Use of Select Mortality Tables*); vol. xl., p. 302 (Actuarial Note, No. 3, *Law of Uniform Seniority* ; vol. xli., p. 97 (G. J. Lidstone, *Value of a Complete Annuity*); vol. xliii., p. 99 (W. P. Elderton, *Value of a Complete Annuity*); vol. xliv., p. 402 (G. J. Lidstone and S. E. Macnaghten, *Integration by Parts*).

Transactions of the Faculty of Actuaries, vol. v., p. 130 (W. Borland, *Differential Coefficients*).

Little need be said about Part I., as the Algebra and Calculus are learnt from text-books of a kind to which all students have become accustomed in their school work, but in working at the Calculus they should remember that its application in actuarial work is mainly to statistics so that their work will ultimately have to be expressed in arithmetical results. This means that we must be able to find arithmetical values for the differential coefficients and integrals of various functions, the algebraic form of which is unknown or, if known, is such as not to lend itself to direct integration. In practice this implies that both the differential coefficient and the integral of $f(x)$ must be expressed in terms of

$$f(x),\ f(x+a),\ f(x-a),\ f(x+b),\ \text{etc.},$$

* *Journal* of the Institute of Actuaries.

because if, as usually happens, we can find arithmetical values of $f(x)$ for various values of x, we can then reach approximate arithmetical values for the differential coefficients and integrals. An example may make the point clearer. Let us assume that all we know about $f(x)$ is given in the following table:—

x	2	3	4	5	6	7	8
$f(x)$	8·7	10·7	13·2	14·1	13·7	13·1	12·3

It is clear that if we wish to find arithmetical values for the integral of $f(x)$ from $x=2·5$ to $x=6·5$, we must find a method of approximation which only necessitates the employment of the given values of $f(x)$. An obvious approximation that will immediately occur to one is obtained by the use of the mid-ordinate—or a series of mid-ordinates—so that we might take

$$10·7 + 13·2 + 14·1 + 13·7 = 51·7$$

as the value of the integral. More accurate results may be obtained by other assumptions, and these will be found described in Text-Book, Part I., ch. ix., and *J.I.A.*, xl., p. 116, section 1.

A similar type of approximate work is implied in expressing the value of the integral of $f(x)$ from 0 to n in terms of the series—

$$f(0) + f(1) + f(2) + \ldots\ldots + f(n)$$

and of the differences or differential coefficients of $f(o)$ and $f(n)$. A modification of this arises when we have to find a value for

$$f(0) + f(\tfrac{1}{m}) + f(\tfrac{2}{m}) + \ldots.$$

The methods of dealing with these two problems are by the use of either the Euler-Maclaurin Expansion or Lubbock's formula (*see* T.B. I., chap. ix., and T.B. II., chap. xxiv., section 21, et seq.). The former is often known in actuarial circles by the name of Woolhouse who introduced its use there.

When working at finite differences, the student should understand that his aim is to reach an arithmetical result and that the algebra is merely a means to this end. The results he will want are:—

(1) The value of $f(r)$ where r is frequently fractional when he knows $f(a), f(a+1), f(a+2)$, etc.

(2) The value of $f(r)$ when $f(a)$, $f(b)$, $f(c)$, etc., are known, a, b, c, etc., not being equidistant.

(3) The values of $f(a+1)$, $f(a+2)$, etc., $f(a+b+1)$, $f(a+b+2)$, etc. (*i.e.*, the complete series), when only $f(a), f(a+b), f(a+2b), f(a+3b)$, etc., are known.

(4) The sums of series $f(a), f(a+b)$, etc.

Occasionally interpolated values are required when there are two variables.

Broadly speaking, all interpolated results are obtained by assuming that $f(x)$ can be expressed in the form—

$$a+bx+cx^2+dx\ +ex^4,\text{ etc.},$$

and if the student remembers this underlying fact and uses straightforward methods he will have little difficulty; but he must do a number of numerical examples or he will never learn the subject in the practical way in which he will have to use it.

When the student comes to Section B of Part I.. he will be tempted to learn formulæ, and if he falls into this temptation, or if he contents himself with playing about with algebraic expressions, he will never get at the real meaning of the subject. The best method is to read the algebraic treatment and then use the methods to work out arithmetical results, and learn how to use the interest tables at the end of the Text-Book. Unless and until he can turn everything quickly and easily into an arithmetical result, he cannot know the subject; it

is insufficient to reach the end of the algebra and say, "and then we put in the values for j and m, and get the required arithmetical result"; it is only by suffering the arithmetic that he will appreciate the uselessness of any other method of study. Perhaps it may be forgiven to us if we add that the more mathematical the student the greater is his difficulty, apparently, in facing the necessary arithmetical drudgery.

We may now turn to Part II., and for the sake of students who have to study the subject without a tutor, or wish to start reading in advance of their classes, we give in the next section an outline of reading which indicates the parts of the Text-Book (Part II.) specially requiring attention. It must be remarked that when the Text-Book was prepared, the examination syllabus was materially different from what it now is, as students were not examined in the Calculus or its application till they came to the final examination, and a text-book had to be available in which the Calculus could be avoided. Nowadays the easy line of approach to the subject through the Calculus is available in the first instance, and the notes may help to show how this line of approach can be followed most usefully.

A note on Select Tables is added, and we are glad to be able to reprint in abstract Dr. Sprague's paper on the "Use of Select Mortality Tables." We thank him and the Institute of Actuaries for kind permission to reprint.

II. NOTES ON READING TEXT-BOOK (PART II).

In Chapter i. the student should define L_x as $\int_0^1 l_{x+t}dt$ and m_x as d_x/L_x. He should take the table at the end of the book and see what errors arise from using the usual approximations instead of the exact expressions, bearing in mind that the column headed L_x is the approximate, not the

true, value. He will have to use a formula of approximate summation, such as one of those given in *J.I.A.* xl., p. 116, in order to obtain the more accurate values for comparison.

In Chapter ii. an alternative proof for the probability that exactly r lives out of m will survive a year can be obtained by writing down the fundamental expression—

$$p_{wx} \cdots\cdots_{(r)} (1-p_y)(1-p_z) \ldots\ldots (m-r) \textit{ factors.}$$
$$+p_{wy} \cdots\cdots_{(r)} (1-p_x)(1-p_z) \ldots\ldots (m-r) \textit{ factors.}$$
$$+ \textit{etc.}$$

and then equating it to

$$A_r Z^r + A_{r+1} Z^{r+1} + \ldots\ldots + A_m Z^m.$$

Where A_r, A_{r+1}, etc., are numerical coefficients independent of w, x, etc., and Z has the meaning explained in the Text-Book. In order to find the A's we can assume all the ages equal, and since there are ${}_mC_r$ lines in the fundamental expression it becomes ${}_mC_r\, p^r (1-p)^{m-r}$ which may be expanded and equated to

$$A_r p^r\, {}_mC_r + A_{r+1} p^{r+1}\, {}_mC_{r+1} + \ldots\ldots + A_{r+s} p^{r+s}\, {}_mC_{r+s} + \ldots\ldots$$

Equating coefficients of p^{r+s} the value of A_{r+s} is obtained immediately and consequently the whole series of A's is known.

In dealing with the force of mortality, it is best to start from the definition in formula (19) and then make approximations to the differential coefficient in the ordinary way.

The expectation of life (Chapter iii.) can be left over until after annuities have been read; "expectation" is a special case of annuities which is reached when $i=0$, and when the chapter is subsequently read, the student will start with an exact expression $\frac{1}{l_x}\int_0^\infty l_{x+t}\, dt$ and will reach his approximations to this expression by applying Lubbock's or Woolhouse's formula directly. Consequently he will not read section 24, but will reach the results of it as a simple exercise.

In Chapter iv. the first thing to be done is to express the chances in the form of integrals, and if the student can satisfy himself that the chance of x dying before y in the n^{th} year is equal to $\frac{1}{l_x\, l_y}\int_{n-1}^{n} l_{x+t}\, \mu_{x+t}\, l_{y+t}\, dt$, and if he can see the ordinary approximations that result therefrom by assuming that the integral is approximately given by the mid-ordinate, he will be able to save himself from reading sections 1—8 and 11, he will find that formula (14) comes directly from an integral, and will neglect sections 14, 17, 18, 26 and 27.

It may however be remarked that approximate expressions for nearly all survivorship probabilities can be written down easily by assuming the deaths at a particular age to occur in the middle of that year of age, *e.g.*, the probability of x dying in the n^{th} year, y having died before him, and leaving z surviving him is thus $d_{x+n-1}\, l_{z+n-\frac{1}{2}}\, (l_y - l_{y+n-\frac{1}{2}}) \div l_x l_z l_y$. This method is simpler and not necessarily less accurate than the traditional assumption of an even distribution of deaths. The result shown in formula (14) is obtained at once by deducting from unity (which is the total chance that x dies before or after y) the value of the probability that x will be alive t years after y's death. Express this latter probability as an integral.

Chapter v. should be studied by taking each example and working out the result by common sense from first principles; there is no need to read any part of the chapter unless a mistake is made in the attempt to solve the problem. The matter in sections 18—21 provides a useful example of the use of the double integral (*see J.I.A.* xliv., p. 403, formula (2)). The identity of $\int_0^\infty \mathrm{T}_{x+t}\, dt$ with $\int_0^\infty t l_{x+t}\, dt$ is at once seen. Sections 18—69 of chapter vi. need not be read, as they deal with graduation.

Chapter vii. is most important because it introduces interest as well as probabilities. A helpful rule of action in dealing

with problems on annuities is to attempt the solution in terms of probabilities and interest and convert subsequently into commutation columns. Section 98 can be proved in the following alternative way:—

Assume death to occur in the n^{th} year (the chance of which is $d_{x+n-1} \div l_x$) then the payments of the annuity-due accumulated to the end of that year at j amount to

$$(1+j)\,\frac{(1+j)^n-1}{j}$$

and the present value at i is

$$(1+i)^{-n}\,\frac{(1+j)^n-1}{j}\,(1+j)$$

The solution of the problem is therefore

$$\sum_{n=1} (1+i)^{-n}\,(i+j)\,\frac{(i+j)^n-1}{j}\,\frac{d_{x+n-1}}{l_x}$$

which reduces at once to the expression (68).

Sections 104—109 should be omitted.

Chapter viii. is important; to convince himself of it we recommend the student to find Assurance Values or Annual Premiums from annuities by means of conversion tables and compare the work with what is required when they are calculated by means of commutation columns or by other methods. One object of conversion tables is to save the calculation of tables; if good conversion tables are available, it is frequently sufficient to tabulate annuity values only.

Our recommendation for chapter ix. is as follows:—Read sections 1—3; omit 4—12; reach formula (11) in section 13 by seeing directly that a life annuity is a perpetuity *less* a perpetuity-due deferred till after the death of the person on whose life the anuuity is to depend; this is an exact result and does not assume an even distribution of deaths; make use of the proof of the approximate relation between A and $A^{(m)}$ given in formula (10); read sections 14 to 18; do sections

19—24 and 26 by starting from the formula at the beginning of section 19 and applying Lubbock's and Woolhouse's formulæ as a matter of routine; read sections 25 and 27 to end. An alternative to section 34 is to interpolate for the result between (1) $1 + a_x$, (2) a_x, (3) $vp_x\, a_{x+1}$, (4) $v^2\, {}_2p_x\, a_{x+2}$, etc.

In chapter x., sections 1—9, 13, 14, 15, 17, 18—21, should be read; formula (7) follows directly from (11) of chapter ix. by putting m equal to infinity. The best way of attacking many problems in assurance is to write down the integrals giving the exact result and approximate to its value.

For chapter xi. read the first four and last three paragraphs and *J.I.A.* xliii., p. 99.

In reading chapter xii. it should be borne in mind that in practice we should not use Lubbock's formula for approximating to the value of a joint life annuity (section 54) although it is well to see how the work could be done. We should write down the integral as is done in section 55, and approximate to it by using a formula of approximate summation. This method gives the value of a continuous annuity, but the necessary adjustment can easily be made if an ordinary annuity be wanted. The student should try the same example with two other formulæ, using different intervals; the larger the interval the less the work, but also the less the accuracy! The same exercise should be performed for sections 57 and 58; it will show the advantage of formulæ of approximate summation and teach their application at the same time.

We would urge the student to think of assurances accurately and therefore to put them, in complicated cases, immediately into the form of integrals. Thus in chapter xiii.

$$\bar{A}^{1}_{xy} = \int_0^\infty v^t\, {}_tp_{xy}\, \mu_{x+t}\, dt$$

$$A^{1}_{xy} = \Sigma v^t\, {}_{t-1|}q^{1}_{xy}$$

where the probability of death has, strictly speaking, to be expressed in the form of an integral as indicated in our note on chapter iv. We do not think a student need read sections 5—13, 24, 33—37, at any rate on a first reading; these approximations are awkward in practice. The result shown in formula (18) is found directly by deducting from A_x (which is the value of 1 payable on the death of x whether before or after that of y) the value of 1 payable on the death of x should he survive the death of y by t years. Express this latter assurance as an integral. The examples of approximate summation may be studied as indicated in our note on chapter xii.

In chapter xiv. omit sections 23—29, 31—33, at any rate in the first instance.

Chapter xv. will be read and examples worked out on the lines already indicated.

Chapter xvi. should be treated as a series of examples on chapter vii., etc., and not as book work. The solution can often be made easily by setting out everything in terms of probabilities and then converting them into commutation columns: beginners often have a difficulty in working directly in this rather artificial medium.

Chapter xvii. requires no comment.

Chapter xviii. often gives difficulty: sections 1—32 relate to policy values and the ways in which they are built up. Sections 33—44 deal with a method of valuation no longer in use, but as is shown in sections 45—8 the results obtained can be used for examining the effects of changes of mortality on policy values, and this matter is further treated in sections 49—73. The next two paragraphs relate to construction of policy-value tables, and then we come to paragraphs which up to section 100 deal with values for fractional periods. These are hard to

follow and the student will find an easy treatment of this part of the subject in the Students' Journal, Vol 1, No. 3, page 24. The remainder of the chapter requires no comment.

Chapter xxi. requires no comment except that, as the subject is essentially practical, the student who is not familiar with calculating machines would do well to take an early opportunity of gaining experience of them.

Chapter xxiv., sections 21 to end, will be read, and the examples of the application of the various formulæ should be noted. An alternative method of obtaining the formulæ will be found in J.I.A. xl., page 116, a paper recommended for reading in connection with Part I.

We conclude these notes by remarking that no method, formula, or approximation is worthy of much trouble unless it can be used conveniently in arithmetical work. The student must never forget that the object of all algebraical analysis is to lead to easier methods of producing arithmetical results.

NOTES ON SELECT MORTALITY TABLES.

A Select Mortality Table is a development of the ordinary mortality table showing the effect upon mortality functions at each age of the time elapsed since commencement of observation.

The reason why we use Select Tables is that it has been found by observation that lives chosen (or selected) by medical examination or otherwise, are subject to rates of mortality different from those experienced by the general body of lives of the same age. Therefore, in the interests of accuracy, we are bound to consider time since entry as well as age.

The student is familiar with the notation of an ordinary Mortality Table, and we shall now explain how that notation is adapted to the case of a Select Table. The underlying

principle is the division of the simple suffix $x+t$ (which is the age attained at the point considered) into two suffices $[x]$ and t, where $[x]$ is the age at entry (indicated always by a square bracket) and t is the time elapsed since entry.

Thus $q_{[x]+t}$ is the rate of mortality among lives now aged $x+t$ who were selected at age x, t years ago, while q_{x+t} is the rate of mortality among lives now aged $x+t$ who were selected at various ages and have been under observation for different periods.

(Similar principles hold with other functions *e.g.*, A_{x+t} becomes $A_{[x]+t}$, ${}_nV_{x+t}$ becomes ${}_nV_{[x]+t}$, and so on).

As an example let us suppose that we have before us a pile of cards each of which relates to a life assured whose present age is 30.

If we observe the whole group for one year we shall obtain q_{30}, *i.e.*, a rate of mortality dependent purely upon the age attained.

If, however, we sort out all those cases where the age at entry was 29 (the time elapsed being therefore one year) we shall obtain $q_{[29]+1}$, *i.e.*, a rate of mortality which takes account of age and time.

Similarly by taking out the cards relating to lower ages at entry we shall obtain $q_{[28]+2}$, $q_{[27]+3}$, $\dots q_{[20]+10}$, and so on.

The following table shows the actual figures according to the British Offices $O^{[M]}$ Table :—

$$
\begin{aligned}
q_{[30]} &= \cdot 00312 \\
q_{[29]+1} &= .00493 \\
&\vdots \\
q_{[25]+5} &= .00651 \\
&\vdots \\
q_{[20]+10} &= \cdot 00757
\end{aligned}
$$

It will be noticed that, although in all cases the age attained is 30, the rates of mortality vary. The variations are due to

time elapsed since selection took place and consequently the rates are termed Select Rates.

In theory the influence of time could be traced for an indefinite number of years, but in practice it is found that only the first 10 years need be considered, and, in fact, the student will find that many tables trace the effect of selection for only 5 years, *e.g.*, the British Offices experience for non-profit assurances included in the tables used in the examinations. There is much practical convenience in thus limiting the number of years when it is permissible to do so. With the ordinary mortality table we have one set of functions for each age. With a select table we have 6 sets if we trace selection for 5 years, and 11 sets if we trace it for 10 years, *i.e.*, we have to prepare in effect 6 and 11 mortality tables instead of 1 table.

The student must be thoroughly clear as to the meaning of three actuarial terms, viz., Select, Aggregate and Ultimate.

Select is the term used when we wish to allow for the two factors of age at entry and time elapsed since entry.

The suffices are of the type above described, viz. $[x]+t$. Thus we have $q_{[x]+t}$, $M_{[x]+t}$, etc.

Aggregate is the term used when we wish to allow only for age attained. The suffices are of the type $x+t$. Thus we have q_{x+t}, M_{x+t}, etc.

Ultimate is the term used when we wish to deal with select data but to exclude from them the early portions during which we distinguish both age and time.

Thus the $O^{[M]}$ Ultimate Table is the experience of those lives who formed the data for the Select Table excluding, however, their select period which, as we explained above, is 10 years.

Example. q_{40}, which is the "ultimate" rate of mortality at age 40, is derived from the observation of lives who entered at all

ages up to (but not beyond) 30. Consequently all the effects of the selection at entry are supposed to be eliminated in the Ultimate Table.

The suffices are of the type $x+t$, and whenever ultimate mortality is referred to an indication to that effect is given.

Note on the arrangement of Select Tables. There are two ways of arranging these tables and it is imperative that the student shall distinguish carefully between them.

Modern practice adopts the following form (selection is assumed to be traced for 5 years):—

Age at Entry $[x]$	$q_{[x]}$	$q_{[x]+1}$	$q_{[x]+2}$	$q_{[x]+3}$	$q_{[x]+4}$	q_{x+5}	Age Attained $x+n$
20							25
21							26
22							27
.							.
.							.
.							.

Dr. Sprague's practice was as follows:—

Age x	$q_{[x]}$	$q_{[x-1]+1}$	$q_{[x-2]+2}$	$q_{[x-3]+3}$	$q_{[x-4]+4}$	q_x	Age x
20							20
21							21
22							22
.							.
.							.
.							.

In the former example as we read horizontally the age attained, $[x]+t$, increases by 1 per column.

In the latter case the age attained remains constant.

Key to the Notation used in the following Paper.

The subscripts $_{[x]}$, $_{[x+1]}$, ... $_{[x+n]}$, etc., denote that the symbols to which they are attached relate to lives which are respectively of the ages x, $x+1$, ... $x+n$, etc., and are select (or healthy) lives.

The subscript $_{[x]+n}$ denotes that the lives to which the symbol relates are of the age $x+n$, and were select at the age x.

The subscripts $_{[x-1]+1}$, $_{[x-2]+2}$, ... $_{[x-t]+t}$, denote that the lives are now of the age x, and were select at the ages $x-1$, $x-2$, . . . $x-t$, respectively.

The letter h prefixed to a symbol, as in $(hl)_x$, $(hd)_x$, $(h\mathbf{a})_x$, denotes that the lives to which the symbol relates are select (or *h*ealthy).

The symbols $(hl)_{[x]+n}$, $(h\mathbf{a})_{[x]+n}$, etc., relate to lives which were select at the age x, and are now of the age $x+n$, and still select.

The same is the case with $_n(hV)_x$ or $_n(hV)_{[x]}$.

The letter u prefixed to a symbol denotes that the lives to which it relates are the damaged (or *u*nhealthy) lives of our table. When the subscript is $_{[x]+n}$, the lives were select at the age x.

THE USE OF SELECT MORTALITY TABLES.

Abstract of the last part of a Paper by DR. T. B. SPRAGUE *appearing in* Journal of the Institute of Actuaries, *Vol.* xxii., *p.* 407. (*Date : January,* 1881.)

By means of these tables we may answer with more or less accuracy a variety of questions to which hitherto no numerical answers could be given : in particular, we are able to investigate in a complete manner the effect upon life insurance finance of the gradual wearing out of the benefit of selection. As is well known, a body of insured lives who are at the outset all select, will, after the lapse of some years, contain a number of damaged lives, the majority, however, being still good lives. For some purposes, as for instance, when we are estimating the liabilities of a life insurance company, we may, without inconvenience or error, proceed as if all lives of the same age and standing were equally deteriorated ; but in other cases, as for instance, when we have to fix a rule for the calculation of surrender values, it is important to bear in mind the real character of the deterioration which has taken place, namely, that, while the majority of the lives still remain in good health, a certain small proportion of them have become more or less diseased. It is no new suggestion that the lives which apply to surrender their policies, will, on the average be in a better state of health than those which continue their policies ; that those who surrender should therefore be treated as select lives, and the values of their policies calculated accordingly. It is obvious that while, by the use of such a table as the $H^{M(5)}$, we may calculate *average* policy-values, that shall be applicable to the policies taken as a whole, yet the liability in respect of a policy on a life that has become damaged, must be considerably greater than this average, and therefore the liability under a policy on a life which is still select must

be less than the average; but I am not aware that any attempt has ever been made hitherto to calculate the different values of policies on select and damaged lives, or the respective numbers of healthy and damaged lives contained in a body of mixed lives that have been assured for any length of time. It is my object to do this in the present paper.

In the first place, it will be convenient to consider the net premiums given by my tables. It will be found that up to the age of 43 my net premiums are greater than the H^M premiums, and above that age they are less.

Table of H^M 4 *per-cent net Premiums for the Insurance of* 100.

Age at Entry.	Aggregate.	Select.
15	1·052	1·188
20	1·245	1·391
25	1·428	1·509
30	1·669	1·714
35	1·969	1·992
40	2·352	2·361
45	2·865	2·851
50	3·542	3·488
55	4·458	4·358
60	5·715	5·541
65	7·427	7·138
70	9·866	9·353
75	13·299	12·163

We have next to consider the average policy-values resulting from my tables. The formula for the net-premium value of a policy taken out n years ago at the age x, will be

$$A_{[x]+n} - \mathbf{a}_{[x]+n}.\pi_{[x]} = \mathbf{a}_{[x]+n}(\pi_{[x]+n} - \pi_{[x]}) = 1 - \frac{\mathbf{a}_{[x]+n}}{\mathbf{a}_{[x]}};$$

and when n is greater than 4, these formulas become

$$A_{x+n} - \mathbf{a}_{x+n}.\pi_{[x]} = \mathbf{a}_{x+n}(\pi_{x+n} - \pi_{[x]}) = 1 - \frac{\mathbf{a}_{x+n}}{\mathbf{a}_{[x]}},$$

where the functions A_{x+n}, $\mathbf{a}_{x+n}$, π_{x+n}, are all taken from the

$H^{M(5)}$ table. If in the second set of formulæ we substitute instead of $\pi_{[x]}$ and $\mathbf{a}_{[x]}$, the premium and annuity-due of the H^M table, we shall get the values of policies as found from the combined H^M and $H^{M(5)}$ tables; and this shows us that the policy-values as found from my tables, will be greater or less than those found from the combined H^M and $H^{M(5)}$ tables, accordingly, as $\pi_{[x]}$ is less or greater than the net premium deduced from the H^M table, that is, up to age at entry 43 the policy-values given by my tables are less than those given by the combined H^M and $H^{M(5)}$, and above that age they are greater.

Column (4) of the following tables shows the 4 per-cent policy-values calculated by these formulas, for ages at entry 30, 45 and 60, and for values of n, 1, 2, 3, ... 10, 15, 20, 30: and for the purpose of comparison column (2) of the same tables shows the 4 per-cent. values calculated by the H^M table, and column (3) the values calculated by the combined H^M and $H^{M(5)}$ tables. It will be noticed that there are no values opposite the first four ages in column (3). In practice the blank would be supplied by taking the values from column (2), since in valuing policies under five years' standing the H^M table would be used alone. An examination of the table shows that for such policies, whatever the age at entry, my policy-values greatly exceed the H^M values. Combining this result with the one above, it seems probable that the adoption of my tables would have the effect of considerably increasing the estimated liability of a company under its policies. I have, however, made no calculations as to this; and being at the present time too fully occupied to do so, I shall be pleased if some other member of the Institute will investigate the point.

Values of Policies for 100 *on Mixed and on Healthy Lives, calculated at* 4 *per cent. Interest according to different formulæ.*

Years elapsed.	Mixed Lives.			Healthy Lives.	Mixed Lives.			Healthy Lives.	Years elapsed.
	$100{}_{n}V_{x}$			$100{}_{n}(hV)_{x}$	$100{}_{n-1}V_{x+1}$	$100V^{*}_{([x+1],[x]+n)}$		$100{}_{n-1}(hV)_{x+1}$	
		Combined $H^{M(5)}$ and				Combined $H^{M(5)}$ and			
n	H^{M}	H^{M}	Select.	Select.	H^{M}	H^{M}	Select.	Select.	n
(1)	(2)	(3)	(4)	(5)	(6)	(7)	(8)	(9)	(10)
Age at Entry, 30.									
1	·971	..	1·339	·921	·000	..	·421	·000	1
2	1·970	..	2·480	1·825	1·009	..	1·573	·912	2
3	2·999	..	3·535	2·771	2·047	..	2·638	1·867	3
4	4·060	..	4·579	3·738	3·119	..	3·692	2·843	4
5	5·152	6·420	5·656	4·746	4·222	5·502	4·779	3·861	5
6	6·272	7·487	6·732	5·793	5·352	6·580	5·865	4·917	6
7	7·416	8·577	7·831	6·883	6·507	7·680	6·974	6·017	7
8	8·583	9·690	8·952	8·018	7·687	8·804	8·106	7·162	8
9	9·780	10·826	10·098	9·197	8·895	9·952	9·262	8·353	9
10	11·010	12·011	11·293	10·423	10·137	11·148	10·468	9·590	10
15	17·817	18·712	18·048	16·976	17·011	17·915	17·287	16·204	15
20	25·343	26·158	25·555	24·185	24·611	25·433	24·862	23·480	20
30	42·314	42·755	42·287	40·765	41·748	42·193	41·751	40·215	30
Age at Entry, 45.									
1	1·782	..	2·341	1·658	·000	..	·694	·000	1
2	3·585	..	4·464	3·351	1·836	..	2·853	1·721	2
3	5·413	..	6.492	5·091	3·697	..	4·916	3·491	3
4	7·270	..	8·440	6·864	5·587	..	6·896	5·293	4
5	9·157	10·149	10·333	8·683	7·509	8·519	8·821	7·144	5
6	11·085	12·028	12·209	10·542	9·472	10·433	10·728	9·033	6
7	13·059	13·940	14·116	12·438	11·481	12·379	12·668	10·961	7
8	15·070	15·897	16·070	14·367	13·529	14·372	14·655	12·923	8
9	17·111	17·877	18·046	16·344	15·607	16·387	16·664	14·933	9
10	19·180	19·895	20·059	18·362	17·714	18·441	18·711	16·986	10
15	29·808	30·344	30·487	28·654	28·534	29·080	29·314	27·451	15
20	40·470	40·862	40·983	39·027	39·390	39·789	39·988	37·999	20
30	60·856	61·076	61·156	58·166	60·146	60·370	60·501	57·460	30
Age at Entry, 60.									
1	3·066	..	4·386	2·867	·000	..	1·564	·000	1
2	6·125	..	8·157	5·749	3·155	..	5·447	2·967	2
3	9·168	..	11·405	8·706	6·295	..	8·790	6·011	3
4	12·186	..	14.392	11·703	9·408	..	11·866	9·097	4
5	15·190	15·748	17·281	14·539	12·507	13·083	14·839	12·017	5
6	18·188	18·694	20·173	17·398	15·600	16·122	17·817	14·960	6
7	21·188	21·637	23·062	20·274	18·695	19·158	20·792	17·920	7
8	24·198	24·604	25·975	23·172	21·800	22·219	23·790	20·905	8
9	27·236	27·589	28·906	26·052	24·934	25·298	26·808	23·870	9
10	30·271	30·602	31·865	28·878	28·065	28·407	29·854	26·779	10
15	44·234	44·547	45·556	41·365	42·470	42·793	43·949	39·634	15
20	55·978	56·139	56·937	..	54·585	54·752	55·666	..	20
30	74·148	74·538	75·001	..	73·330	73·732	74·263	..	30

* N.B.—For an explanation of this symbol *see* p. 25. [Eds.]

I have stated above how the values in columns (2), (3) and (4) are got, and a full explanation of the values in the other columns will be given presently; but it may be useful to give here a short description of the various formulas by which the different values are calculated.

As already explained, the values in columns (2), (3), (4), are all got by the well-known formulas.

$$_{n}V_{x}=A_{x+n}-\mathbf{a}_{x+n}.\pi_{x}=\mathbf{a}_{x+n}(\pi_{x+n}-\pi_{x})=1-\mathbf{a}_{x+n}\div\mathbf{a}_{x}.$$

In column (2) the values of all the quantities are taken from the H^{M} Table.

In (3), n being >4, the values of A_{x+n}, $\mathbf{a}_{x+n}$, π_{x+n}, are taken from the $H^{M(5)}$ Table, but those of $\mathbf{a}_{x}$ and π_{x} from the H^{M} Table.

In (4) the formulas become

$$_{n}V_{[x]}=A_{[x]+n}-\mathbf{a}_{[x]+n}\cdot\pi_{[x]}=\mathbf{a}_{[x]+n}(\pi_{[x]+n}-\pi_{[x]})=1-\mathbf{a}_{[x]+n}\div\mathbf{a}_{[x]};$$

and the values of $A_{[x]+n}$, $\mathbf{a}_{[x]+n}$, $\pi_{[x]+n}$, are taken from my Select Tables when $n<5$ and from the $H^{M(5)}$ Table when $n=$or>5; the values of $\mathbf{a}_{[x]}$ and $\pi_{[x]}$ being always taken from the select Tables.

In (5) the life is supposed to be still select (or healthy) at the date of valuation, and the formulas are

$$_{n}(hV)_{[x]}=A_{[x+n]}-\mathbf{a}_{[x+n]}.\pi_{[x]}=\mathbf{a}_{[x+n]}(\pi_{[x+n]}-\pi_{[x]})=1-\mathbf{a}_{[x+n]}\div\mathbf{a}_{[x]},$$

the values of all the quantities being taken from my Select Tables. Since a life is always supposed to be select at entry, we may omit the brackets round the age at entry x, here and above, and write without any risk of confusion, $_{n}V_{x}$ and $_{n}(hV)_{x}$.

Passing now to the second set of values—those in columns (6) to (9), we suppose that the first year's premium has been absorbed by the initial expenses and the first year's risk. The values in (6) and (7) are got by the formulas

$$A_{x+n}-\mathbf{a}_{x+n}.\pi_{x+1}=\mathbf{a}_{x+n}(\pi_{x+n}-\pi_{x+1})=1-\mathbf{a}_{x+n}\div\mathbf{a}_{x+1}.$$

In (6) the values of all the quantities are taken from the H^{M} Table.

In (7), n being > 4, the values of A_{x+n}, $\mathbf{a}_{x+n}$. π_{x+n}, are taken from the $H^{M(5)}$ Table, and those of $\mathbf{a}_{x+1}$ π_{x+1}, from the H^M Table.

In (8) the policy-values are calculated by the new formulas,

$$A_{[x]+n}-\mathbf{a}_{[x]+n}.\pi_{[x+1]}=\mathbf{a}_{[x]+n}(\pi_{[x]+n}-\pi_{[x+1]})=1-\mathbf{a}_{[x]+n}\div\mathbf{a}_{[x+1]}.$$

It is desirable to have a symbol to denote this value, and as the ordinary symbol for a policy-value ${}_nV_x$, does not admit of being suitably modified, I propose $V([x+1], [x]+n)$; consistently with which we might have the ordinary policy-value denoted by $V(x, x+n)$.

In (9), as in (5), the life is supposed to be still healthy, and the policy-value is calculated by the formulæ

$$\begin{aligned}{}_{n-1}(hV)_{[x+1]}=A_{[x+n]}-\mathbf{a}_{[x+n]}.\pi_{[x+1]}&=\mathbf{a}_{[x+n]}(\pi_{[x+n]}-\pi_{[x+1]})\\&=1-\mathbf{a}_{[x+n]}\div\mathbf{a}_{[x+1]},\end{aligned}$$

in which the values of all the quantities are got from the Select Tables. This value might, if desired, be denoted by the symbol $V([x+1], [x+n])$. It will be noticed that in the case of a healthy life, the value obtained in this way is the same as that calculated by the formula of column (5) for a policy on a life still healthy, which was taken out at the age $x+1$ and had been in force n—1 years; but a similar relation does not exist between the policy-values for mixed lives.

I submit that by means of my new tables we can estimate the liability of a life office under its policies more correctly than has ever been possible hitherto. When the *Institute* tables were published, it seemed very probable that we should get approximately true results by using the H^M table for policies of less than five years' duration, and the combined H^M and $H^{M(5)}$ tables for policies of greater duration. On the assumption that after the lapse of five years the effect of selection has worn off, it was clear that for policies of five years or more duration, the $H^{M(5)}$

annuities and reversions were the proper ones to use; but what was the proper net premium? In practice the H^M premium was adopted, but this was only because no more accurate premiums were available. It could not be maintained that the H^M premiums were the correct ones; in fact, it was impossible to say with certainty whether the correct net premiums would be greater or less than these. As regards policies of less than five years' standing, the uncertainty was still greater, as it could not even be known whether the H^M reversions and annuities were greater or less than the correct values when proper allowance was made for the effect of selection. The only legitimate conclusion that could be drawn was that, as the mortality in the early years of insurance is light, probably the reserve that should be made for policies quite recently effected, ought to be greater than that given by the H^M table. All these doubts and difficulties are removed by the use of my Select Tables. We are able to calculate from them for the first time the proper reserve to make for the liability under policies on recently selected lives of any age, when allowance is made for the gradual wearing out of the benefit of selection, and we altogether get rid of the awkward break in the series of policy-values between the fourth and fifth years, which is caused by the use of the combined H^M and $H^{M(5)}$ tables. My tables also, as I shall show further on, enable us to make proper allowance for the initial expenses of the new business.

On these grounds I ask my professional brethren to give a friendly reception to my tables, and to consider for themselves, first, whether these tables are, as I hold, so great an improvement on all previously existing tables, that they deserve to be employed in all life insurance calculations; and then, how they may best be practically applied in making life office valuations. There will be no difficulty in using them for this purpose when the values of policies are determined individually by means of

a table; and I cannot help thinking that this method of valuation is at present unduly discredited. There is one very great benefit that attaches to it, the value of which is strikingly exemplified by recent events; and that is, that it most effectually prevents the introduction of any negative policy-values into a valuation. If a table of policy-values were calculated on the basis of my Select Mortality Tables its extent would be precisely the same as that of a table calculated on any other basis, and the use of it would be attended with precisely the same amount of labour. If, however, the policies are valued, as is now more usual, in classes, separate classifications would have to be made of policies of less and more than five years' duration; and policies of less than five years' duration would have to be arranged both according to the age at entry and the endurance, unless it should be found sufficiently correct to group all policies under five years' standing simply according to the age attained, and then to value the policies at any age, say y, by means of the annuities and assurances for the age $[y-2]+2$. I throw out this suggestion in the hope that some member of the *Institute* will take it up, examine it practically, and publish the result of his examination. So many interesting lines of enquiry are suggested by a study of my new tables that I cannot myself undertake to follow them all up. It seems probable, indeed, that a complete examination of them would afford employment for many investigators during a long period; and having laid unreservedly before the actuarial profession my methods of investigation and the results at which I have thus far arrived, I trust other volunteers will now come forward and assist in the exploration of the new region into which I may at all events claim to have opened a practicable road.

I now pass on to consider what is the value of a policy on a life which is still select. Taking the general formula for the net value of a policy, $A_{x+n}-\mathbf{a}_{x+n}.\pi_x$, it is obvious that the fact of

the life being still select is no reason for altering the value of π_x; but the values of the assurance and the annuity must be those for select lives. The formulas for the value of a policy on a select life, which I will denote by ${}_n(hV)_x$, become

$${}_n(hV)_x = A_{[x+n]} - \mathbf{a}_{[x+n]}\pi_{[x]} = \mathbf{a}_{[x+n]}(\pi_{[x+n]} - \pi_{[x]}) = 1 - \mathbf{a}_{[x+n]} \div \mathbf{a}_{[x]};$$

and since $\mathbf{a}_{[x+n]} >$ both $\mathbf{a}_{[x]+n}$ and $\mathbf{a}_{x+n}$, we see that the policy-values given by these formulas will always be less as might have been anticipated, than those given by the others. The values resulting from this formula are given in column 5 of the table, and these values, it will be observed, are the sums that an office should have in hand, according to the net-premium method of valuation, in order to provide for its liability under a policy upon a life that is still select. It is obvious that these values will have a most important bearing on the calculation of surrender values; but since, in calculating them, no allowance is made for the initial expenses, I do not consider them so suitable for practical adoption as those which I shall presently explain.

In all the preceding formulas, I have assumed, in accordance with the fundamental principle of the net-premium method of valuation, that the net premium valued is that for the original age at entry, namely, $\pi_{[x]}$. I am of opinion, however, as I have on various occasions explained, that, having regard to the higher rate of expenditure which necessarily attaches to the first year of a policy, this is not the correct method of procedure. If, as is often the case, the first year's premium is wholly absorbed by the current risk and expenses, I have suggested that, in the formulas for a policy value, ${}_nV_x = \mathbf{a}_{x+n}(\pi_{x+n} - P)$, P should be put $= \pi_{x+1}$, so as to value the net premium for an age one year greater than the age at entry, thus making no reserve for policies of only one year's standing. This process cannot be adopted without some modification when we make use of the new tables. We then have

$$_1V_x = A_{[x]+1} - \mathbf{a}_{[x]+1}.P = \mathbf{a}_{[x]+1}(\pi_{[x]+1} - P)$$

$$_1(hV)_x = A_{[x+1]} - \mathbf{a}_{[x+1]}.P = \mathbf{a}_{[x+1]}(\pi_{[x+1]} - P).$$

These equations show that, if we put $P = \pi_{[x]+1}$, we make $_1V_x = 0$; but, since $\pi_{[x]+1} > \pi_{[x+1]}$, the same substitution gives a negative value for $_1(hV)_x$, which is of course inadmissible. We may, however, put $P = \pi_{[x+1]}$; and we then get

$$_1V_x = \mathbf{a}_{[x]+1}(\pi_{[x]+1} - \pi_{[x+1]}) = 1 - \frac{\mathbf{a}_{[x]+1}}{\mathbf{a}_{[x+1]}};$$

$$_1(hV)_x = \pi_{[x+1]}(\pi_{[x+1]} - \pi_{[x+1]}) = 0.$$

We thus make no reserve for lives which are still select, but, looking at the lives as a whole, we do make a reserve for policies which have been one year in force; and, as we shall hereafter see, this reserve is exactly enough to meet the liability under the policies on the lives that have become damaged in the year.

These results enable us to calculate the value of the option that a policyholder can exercise at the end of the first insurance year. The company, in granting a whole-life policy, enters into a contract which the policyholder is at liberty to continue or discontinue, as he may think proper; and the premiums received by the company in the first year must therefore be sufficient to provide, not only for the heavy initial expenses and for the risk of death during the first year, but also for the risk of the life becoming damaged in the first year, since in that event it is morally certain that the policyholder will continue the policy in force. Let us use the phrase "first year's risk" to include both of the risks just mentioned, and investigate what is the minimum net premium, Q, which must be received from each of $l_{[x]}$ select lives of the age x, in order that this risk may be exactly met. The number of deaths which will occur among the $l_{[x]}$ lives is $l_{[x]} - l_{[x]+1}$, and the number of survivors at the end of a year is $l_{[x]+1}$, for each of whom, as we have seen, must be made a reserve equal to $1 - \mathbf{a}_{[x]+1} \div \mathbf{a}_{[x+1]}$ The total sum, therefore,

which the company must have in hand at the end of the year, to provide for the claims then due and for the policy-values, is

$$l_{[x]} - l_{[x]+1} + l_{[x]+1}.\left(1 - \frac{\mathbf{a}_{[x]+1}}{\mathbf{a}_{[x+1]}}\right) = l_{[x]} - l_{[x]+1}.\frac{\mathbf{a}_{[x]+1}}{\mathbf{a}_{[x+1]}};$$

and this must be equal to the accumulated amount of the $l_{[x]}$ premiums, $=(1+i)l_{[x]}\,\underset{\sim}{Q}$. Hence

$$\underset{\sim}{Q} = v\left(1 - \frac{l_{[x]+1}}{l_{[x]}}.\frac{\mathbf{a}_{[x]+1}}{\mathbf{a}_{[x+1]}}\right) = v\left(1 - p_{[x]}.\frac{1+a_{[x]+1}}{\mathbf{a}_{[x+1]}}\right) = v - \frac{a_{[x]}}{\mathbf{a}_{[x+1]}}.$$

The following table shows the values of this quantity for quinquennial ages at entry, and the values here shown seem deserving of careful study, as indicating what is the real risk run by a life office during the first year's currency of a whole-life policy. It follows that the office can afford to spend only the excess of the premium received over the amount here shown; and that, if it spends more than this, it runs a risk of its new business, resulting in a loss by the dropping of the policies on the healthy lives at the end of the first year.

Age at Entry.	First Year's Risk.	Age at Entry.	First Year's Risk.
15	·002945	45	·012783
20	·010640	50	·015632
25	·009161	55	·020064
30	008360	60	·028662
35	·009450	65	·040568
40	·010258	70	·058932

We may now say that, on the assumption that the first year's premiums are absorbed by the first year's risk and the initial expenses, the proper net premium to value is $\pi_{[x+1]}$; and then, if the expenses in future years are approximately constant, we must value the same net premium for policies of all durations, and we have the formulas

$${}_nV_x = A_{[x]+n} - \mathbf{a}_{[x]+n}.\pi_{[x+1]} = \mathbf{a}_{[x]+n}(\pi_{[x]+n} - \pi_{[x+1]}) = 1 - \frac{\mathbf{a}_{[x]+n}}{\mathbf{a}_{[x+1]}};$$

$${}_n(hV)_x = A_{[x+n]} - \mathbf{a}_{[x+n]}.\pi_{[x+1]} = \mathbf{a}_{[x+n]}(\pi_{[x+n]} - \pi_{[x+1]}) = 1 - \frac{\mathbf{a}_{[x+n]}}{\mathbf{a}_{[x+1]}}.$$

The values in columns (8) and (9) of the table of policy-values are calculated by these formulas: and these are the values which, in my opinion, may most advantageously be used in life office calculations.

The values in column (8) are suitable for calculating policy-values in the quinquennial or other periodical investigations, and a table of them might be formed and used in the same way as the well-known tables of policy-values which are in use in a good many offices. The values in column (9) are useful for calculating surrender values, but must not be employed without some further adjustment. They represent the sum that the company should have in hand to meet its liability under a policy, on the suppositions that the first year's premium is wholly absorbed by the first year's risk and expenses, and that the life is still in good health. It is therefore clear that an office cannot afford to give for the surrender of the policy a larger sum than shown by these values; but it may very properly give a smaller value. I will not, however, stay on the present occasion to consider on what principle this smaller value should be calculated.

In finding the value of a policy on a life that is still select, we have taken the values of $A_{[x]}$ and $\mathbf{a}_{[x]}$, our object being to determine with the greatest attainable accuracy the real value of the sum assured and the premium: but when we bear in mind that the rate of mortality among insured lives depends upon the rate of lapse, in such a way that, the greater the rate of lapse, the greater will be the rate of mortality, we see that, in order to determine the true value of the policy, we ought to know the values of assurances and annuities upon select lives among which there are no withdrawals. At present, however, I believe there are no tables in existence that will give us the values of these quantities. The fact is that, although we have ample materials for determining the exact rate of mortality among

insured lives, and are able to determine with great accuracy the rate of mortality during the first insurance year for entrants of any age, and to trace the gradual increase in the rate of mortality among them caused by the combined operation of advancing age and the withdrawals of healthy lives, yet we have no means at present of determining how the rate of mortality would increase if there were no such withdrawals. All we can say is, that there is good reason for believing that the rate of mortality would be less than is now found to prevail among insured lives, and consequently the values of assurances would be less and the values of annuities greater, than those given by our tables. It follows also that the net premiums for the insurance of such lives would be less than those above calculated; but whether the values of policies would be greater or less is not obvious.

Before the subject of selection can be thoroughly understood, it will be essential to obtain information as to the rate of mortality among select lives, none of which withdraws from observation. Three sources occur to me whence the necessary information might be obtained. One of these I have mentioned in former papers, namely, the experience among the Government annuitants. These annuitants when nominated are probably, on the average, a very select body of lives; and no withdrawals take place among them, since the Government will never repurchase an annuity which it has granted. The second source from which information might be gained is the mortality experience of the Peerage Families. We may fairly assume that men who marry are, taken as a whole, a body of select lives, although there will, no doubt, be occasional exceptions. We might therefore investigate the rate of mortality among the married men of the Peerage Families (supposing each to come under observation at the date of marriage), grouping together all those who marry at the same age, and tracing the rate of mor-

tality in each of these groups. I have myself extracted a very considerable number of facts from the Peerage records, which I may, perhaps, some day utilize in this way. The third source from which information might be obtained, is from the published experience of individual life offices, among which I would particularly indicate the experience of the *New York Mutual*, and the very extensive experience of the *Gotha Life Office*, during 50 years recently published by the Manager, Dr. Emminghaus. If we assume that all the persons who withdraw from observation by the lapse or surrender of their policies, are select lives, and are subject, after withdrawal, to the rate of mortality which we find to prevail among recently selected lives, we might by a long and somewhat complicated series of calculations determine approximately what would be the number of persons under observation in any year, and the number of deaths among them, supposing that no withdrawals took place; and comparison of these would give us the rate of mortality we require.

Thus far we have seen that, after the lapse of any number of years, the survivors of a body of lives originally select, form a mixed body, consisting of comparatively few damaged lives, with a majority of lives still healthy; and that the value of a policy on a life still select, is less than the average value calculated on the assumption that all the lives of the same age, and insured for the same length of time, are in the same state of health. We have also seen how we can calculate both the average values of policies and the values of policies on lives that are still select. It remains to complete this part of the subject by showing how to calculate the number of damaged lives among a body of persons that have been assured for any time, and the values of their policies; and we shall then see that the sum of the values of the policies on the select lives and of the values of the policies on the damaged lives, is equal to

the sum of the average policy-values for the total number of lives.

In the first place it will be convenient to consider lives that have been five years or more insured, and have therefore, according to our fundamental supposition, attained the ultimate rate of mortality for their age. Turning to Table No. 1, at the end, let us in the first instance consider the 862,820 select lives of 30. We see that out of these there survive at the end of 5 years 829,800, who are then of the age 35. Some of these will be still select lives and some will be damaged, and the rates of mortality prevailing among these two classes will of course, be widely different; but the 829,800 will as a whole be subject to the mortality of the $H^{M(5)}$ table. (It is true that according to the table the $H^{M(5)}$ mortality is reached at the age of 33, but this does not in any way affect our argument.) After the lapse of 5 more years, or at the age of 40, the number of survivors will be 786,500. Now according to the construction of the table, this is also the number of survivors out of 820,928 select lives of 35, that is to say, the 820,928 select lives and the 829,800 mixed lives of 35, will give us after 5 years the same number of survivors, who will thenceforward be subject to the $H^{M(5)}$ mortality. It is therefore clear that the difference, 8,872, between these numbers must be the number of damaged lives included among the latter; and consistently with our supposition that the effect of selection wears off in 5 years at most, these damaged lives must all be dead before the end of 5 years, or none of them will attain the age of 40. It will be useful to consider this point a little more closely. When we say that the benefit of selection wears off in 5 years, we mean that after 5 years a body of lives originally select will contain the proportion of damaged lives that will give us the $H^{M(5)}$ rate of mortality; and that after 5 years the survivors of a body of select lives will be subject to the same rate of mortality as the survivors of a body

of mixed lives of the same age; and it is clear that this can only be the case if the damaged lives contained among these mixed lives all die within the 5 years. If we now trace from year to year the number of survivors of the 820,928 select lives and 829,800 mixed lives, as set out for greater distinctness in the following table, it is easy to see that the differences, as shown in column (5), must be the survivors from year to year out of the 8,872 damaged lives, until, when we come to the end of the fifth year, there is no difference, or all the 8,872 are dead.

Age x $=35+t$	Years elapsed t	Survivors of			Number of Damaged Lives dying in a Year $(ud)_{(35)+t}$
		Select Lives $l_{[35]+t}$	Mixed Lives $l_x=l_{35+t}$	Damaged Lives $(ul)_{(35)+t}$	
(1)	(2)	(3)	(4)	(5)=(4)−(3)	(6)
35	0	820,928	829,800	8,872	4,278
36	1	816,906	821,500	4,594	2,304
37	2	810,710	813,000	2,290	1,358
38	3	803,368	804,300	932	726
39	4	795,194	795,400	206	206
40	5	786,500	786,500	0	0

The differences of the numbers in the fifth column will give us the number of damaged lives dying in each year, as shown in the last column. These figures may also be got by means of the numbers dying, as contained in the Tables. Thus we have

Age x $=35+t$	Years elapsed t	Numbers Dying in the following Year.		
		Select $d_{[35]+t}$	Mixed $d_x=d_{35+t}$	Damaged $(ud)_{(35)+t}$
(1)	(2)	(3)	(4)	(5)=(4)−(3)
35	0	4,022	8,300	$4,278=(ud)_{35}$
36	1	6,196	8,500	$2,304=(ud)_{(35)+1}$
37	2	7,342	8,700	$1,358=(ud)_{(35)+2}$
38	3	8,174	8,900	$726=(ud)_{(35)+3}$
39	4	8,694	8,900	$206=(ud)_{(35)+4}$
40	5	8,900	8,900	$0=(ud)_{(35)+5}$

We thus see that, if we denote by $(ul)_x$ the number of damaged (or *un*healthy) lives among the l_x mixed lives, we shall have $(ul)_x = l_x - l_{[x]}$; if, furthermore, $(ul)_{(x)+t}$ denotes the number of these damaged lives that are alive after the lapse of t years, $(ul)_{(x)+t} = l_{x+t} - l_{[x]+t}$; and if $(ud)_{(x)+t}$ denotes the number of these lives that die in the $(t+1)$th year, $(ud)_{(x)+t} = d_{x+t} - d_{[x]+t}$. According to the suppositions we have made, when $t =$ or > 5, $(ul)_{(x)+t} = 0$, $(ud)_{(x)+t} = 0$. Column (8) of Table No. 1 contains the values of $(ul)_x$, or the number of damaged lives calculated in this way; columns (9) to (12) show the numbers of these surviving after 1, 2, 3, 4 years; and columns (13) to (17) show how many of them die in successive years up to the fifth. In each case the numbers are to be read diagonally downwards from left to right.

This table, it is to be observed, puts our adjusted table to a very severe test, which I think it stands on the whole very satisfactorily. It is true that the numbers in columns (11), (12), (16), (17), show considerable irregularities; but these arise from the unsatisfactory graduation of the $H^{M(5)}$ table, and they will affect the money values very slightly.

In what precedes we have shown that, if l_x represents the survivors of a number of lives now of the age x, who have been insured for 5 years or more, then the number of those lives that are still select (or healthy) is $l_{[x]}$, and the number of damaged lives among them, $(ul)_x$, is the difference, $l_x - l_{[x]}$. We have next to determine the number of healthy and damaged lives among a body of lives of any age that have been insured for less than 5 years. In order to fix the ideas, let us first suppose the lives to be now of the age 35, and to have been insured for 1 year. Then Table 1 (p. 52) shows us that 825,510 such lives will give after the lapse of 5 years 786,500 mixed lives; but, from the construction of the table, this is also the number of survivors out of 820,928 select lives of the age 35. Tracing by

means of the figures in Table No. 1, the survivors out of the two bodies as shown in the following table, we get the numbers of these damaged lives who survive after the lapse of 1, 2, 3, 4, 5 years.

Age x	Years elapsed t	Respective number of Survivors after n Years			Number of the Damaged Lives who Die in a Year $(ud)_{([34]+1)+t}$
		$l_{[35]+t}$	$l_{[34]+t+1}$	$(ul)_{([34]+1)+t}$	
(1)	(2)	(3)	(4)	(5)=(4)−(3)	(6)
35	0	820,928	825,510	4,582	2,136
36	1	816,906	819,352	2,446	1,106
37	2	810,710	812,050	1,340	748
38	3	803,368	803,960	592	386
39	4	795,194	795,400	206	206
40	5	786,500	786,500	0	0

We now see that for any age x, $l_{[x]}$ is the number of healthy lives among $l_{[x-1]+1}$ lives who have been insured for 1 year; and the same reasoning will show us that it is also the number of healthy lives among $l_{[x-2]+2}$ lives who have been insured for 2 years, among $l_{[x-3]+3}$ lives who have been insured for 3 years, and among $l_{[x-4]+4}$ lives who have been insured for 4 years; and we saw previously that it is also the number among l_x lives who have been insured for 5 years or more. It will be convenient to have a distinctive notation to denote the various numbers of damaged lives. If we put $(ul)_{[x]+n}$ to denote the number of damaged lives contained in $l_{[x]+n}$ mixed lives who have been insured for n years, we have $(ul)_{[x]+n}=l_{[x]+n}-l_{[x+n]}$; and putting $(ul)_{([x]+n)+t}$ for the number of the above mentioned damaged lives that still survive after t years, all being of the age $x+n+t$; and $(ud)_{([x]+n)+t}$ for the number of them who die in the $(t+1)$th year, that is to say, between the ages $x+n+t$ and $x+n+t+1$, we have

$$(ul)_{([x]+n)+t}=l_{[x]+n+t}-l_{[x+n]+t};$$

and $$(ud)_{([x]+n)+t}=d_{[x]+n+t}-d_{[x+n]+t}=l_{[x]+n+t}-l_{[x]+n+t+1}$$
$$-l_{[x+n]+t}+l_{[x+n]+t+1}=l_{[x]+n+t}-l_{[x+n]+t}-(l_{[x]+n+t+1}-l_{[x+n]+t+1})$$
$$=(ul)_{([x]+n)+t}-(ul)_{([x]+n)+t+1}.$$

Taking a general view now of our extended table of l_x, we see that, since the radix of the $H^{M(5)}$ table is 10,000 persons alive at the age 10, the number at any age x in our column l_x, is the number of survivors out of 1,000,000 select lives of 10, and is also the number of survivors out of $l_{[y]}$ select lives of the age y, where $y =$ or $< x-5$; for instance, 551,600 mixed lives of the age of 60 are the survivors out of 611,313 select lives of 55, or out of 624,683 select lives of 54, or out of 674,923 select lives of 50, or out of 862,820 select lives of 30, and so on. Again, we see that $l_{[x]}$ is the number of lives still select among all the numbers of the same age x in the different columns of our table; and the number of damaged lives among the number of mixed lives in any of those columns, is found by subtracting from it the corresponding number $l_{[x]}$. We further see that $l_{[x]}$ is the number of healthy lives among the survivors of $l_{[x-1]}$ select lives who have been insured for one year, among $l_{[x-2]}$ select lives who have been insured for two years, and in general among $l_{[x-n]}$ select lives who have been insured for n years. If we suppose that the ultimate rate of mortality is not attained after 5 years, but after a longer term, for instance 10 years, we may construct a table with similar properties, which are sufficiently obvious without explanation.

It will now be useful to show by an example the gradual deterioration that takes place among a body of lives originally select. Taking 820,928 select lives of 35, we have seen that the number of damaged lives at the end of a year is $l_{[35]+1} - l_{[36]} = 4{,}568$. The number of select lives among the survivors being $l_{[36]}$, the number of those that become damaged in the next insurance year is obviously $l_{[36]+1} - l_{[37]} = 4{,}506$, the number that become damaged in the following insurance year is $l_{[37]+1} - l_{[38]} = 4{,}460$, and so on; and we have seen above how to ascertain the number of survivors out of these various numbers of damaged lives, after the lapse of any number of

years. The following table shows (in column (2)) the number of the 820,928 lives becoming damaged in each of the first 5 insurance years, and (in columns (3) to (6)) the respective numbers of them surviving, until they are all dead ; and the last column shows the total numbers of these survivors at the end of each of the 10 years from the date of entry. It will be noticed that the number of damaged lives existing at the age 40, and the numbers of them surviving in successive years. agree with the figures given in columns (8) to (12) of Table No. 1.

Age x	$(ul)_{[x-1]+1}$	$(ul)_{([x-2]+1)+1}$	$(ul)_{([x-3]+1)+2}$	$(ul)_{([x-4]+1)+3}$	$(ul)_{([x-5]+1)+4}$	
(1)	(2)	(3)	(4)	(5)	(6)	(7) = (2) + (3) + (4) + (5) + (6)
35	..	..	..	..	..	..
36	4,568	..	..	..	..	4,568 = $(ul)_{[35]+1}$
37	4,506	2,437	..	..	..	6,943 = $(ul)_{[35]+2}$
38	4,460	2,374	1,338	..	..	8,172 = $(ul)_{[35]+3}$
39	4,426	2,328	1,272	560	..	8,586 = $(ul)_{[35]+4}$
40	4,395	2,287	1,223	488	112	8,505 = $(ul)_{40}$
41	..	2,245	1,172	466	0	3,883 = $(ul)_{(40)+1}$
42	..	..	1,119	453	22	1,594 = $(ul)_{(40)+2}$
43	..	..	..	420	22	442 = $(ul)_{(40)+3}$
44	..	..	..	..	33	33 = $(ul)_{(40)+4}$
45	..	..	..	..	..	..

We are now in a position to calculate the increase in the rate of mortality that is produced by the withdrawal of any number of healthy lives. Taking, for instance, the 829,800 mixed lives of the age of 35, we have seen that the number of damaged lives among them is 8,872, and the number of healthy lives, 820,928. The mixed lives, taken as a whole, are subject to the $H^{M(5)}$ rate of mortality; but, as we have seen, the damaged and the select lives considered separately are subject to entirely different rates of mortality. Suppose now that one-tenth of the healthy lives (82,093) withdraw, leaving 738,835 under observation ; then the deaths among the remaining

healthy lives will be reduced proportionately, namely, from 4022 to 3620, while the deaths among the damaged lives will remain as before, namely, 4,278. Adding these together, we get a total of 7,898 deaths out of 747,707 lives, so that the rate of mortality is ·010563, which is about 5 per cent. greater than the $H^{M(5)}$ rate of mortality, ·010002. In the same way we may calculate the rate of mortality in each of the five years until the 8,872 damaged lives are all dead. The rates of mortality thus resulting are contained in column (3) of the subjoined table. If we suppose a larger number of the healthy lives to withdraw. it is clear that the rate of mortality among the lives remaining under observation will be increased, and the greater the proportion of healthy lives that withdraw, the higher will be the rate of mortality. In the following table I have given the rates of mortality that would result, according to our tables, from the withdrawal, not only of 10 per cent., but also of 20, 30, 40 and 50 per cent. of the healthy lives; and it will be seen that the withdrawal of one-half the healthy lives has the effect of increasing the rate of mortality in the following year almost 50 per cent.

Table showing the Rate of Mortality among a body of Lives, aged 35, *that have been insured for* 5 *or more years, on the assumption that a varying proportion of the Healthy Lives withdraw.*

Age.	Proportion of Healthy Lives Withdrawn.						Age.
	0	$\frac{1}{10}$	$\frac{2}{10}$	$\frac{3}{10}$	$\frac{4}{10}$	$\frac{5}{10}$	
(1)	(2)	(3)	(4)	(5)	(6)	(7)	(8)
35	·010002	·010563	·011261	·012157	·013346	·014999	35
36	·010347	·010651	·011031	·011519	·012168	·013073	36
37	·010701	·010883	·011111	·011404	·011793	·012340	37
38	·011065	·011165	·011289	·011448	·011661	·011958	38
39	·011189	011218	·011254	·011300	·011363	·011451	39
40	·011316	·011316	·011316	·011316	·011316	·011316	40

These are calculated by the formula: Probability of dying in the $(n+1)$th year, if $m : 10$ of the healthy lives withdraw at the age $35, = \frac{d_{35+n} - \frac{1}{10} m d_{[35]+n}}{l_{35+n} - \frac{1}{10} m l_{[35]+n}}$. When $n =$ or > 5, $d_{[35]+n} = d_{35+n}$, $l_{[35]+n} = l_{35+n}$; and the formula becomes

$$\frac{d_{35+n}(1 - \frac{1}{10}m)}{l_{35+n}(1 - \frac{1}{10}m)} = \frac{d_{35+n}}{l_{35+n}} = q_{35+n}$$

We are now in a position to show how to calculate the values of annuities and assurances on the damaged lives of our table. Confining our attention, in the first instance, to the $(ul)_x$ damaged lives contained in l_x lives that have been insured for five or more years, and assuming that they are all in the same state of health, we shall have the value of an assurance upon any one of them equal to

$$\left\{ (ud)_x v + (ud)_{(x)+1} v^2 + (ud)_{(x)+2} v^3 + (ud)_{(x)+3} v^4 + (ud)_{(x)+4} v^5 \right\} \div (ul)_x$$
$$= \left\{ (ud)_x v^{x+1} + (ud)_{(x)+1} v^{x+2} + (ud)_{(x)+2} v^{x+3} + (ud)_{(x)+3} v^{x+4} + (ud)_{(x)+4} v^{x+5} \right\} \div (ul)_x v^x$$
$$= (uM)_x \div (uD)_x, \text{ suppose.}$$

This shows us how the columnar method may be extended to the values of benefits dependent upon the damaged lives. Bearing in mind that $(ul)_x = l_x - l_{[x]}$, we see that $(uD)_x = D_x - D_{[x]}$. Also since $(ud)_x = d_x - d_{[x]}$, and $(ud)_{(x)+t} = d_{x+t} - d_{[x]+t}$, we have

$$(uM)_x = d_x v^{x+1} + d_{x+1} v^{x+2} + d_{x+2} v^{x+3} + d_{x+3} v^{x+4} + d_{x+4} v^{x+5}$$
$$- d_{[x]} v^{x+1} - d_{[x]+1} v^{x+2} - d_{[x]+2} v^{x+3} - d_{[x]+3} v^{x+4} - d_{[x]+4} v^{x+5}$$
$$= M_x - M_{[x]},$$

so that we finally get the equation

$$(uA)_x = \frac{(uM)_x}{(uD)_x} = \frac{M_x - M_{[x]}}{D_x - D_{[x]}}$$

Similarly the value of an annuity-due upon a damaged life is

$$\{(ul)_x + (ul)_{(x)+1}v + (ul)_{(x)+2}v^2 + (ul)_{(x)+3}v^3 + (ul)_{(x)+4}v^4\} \div (ul)_x$$
$$= \{(ul)_x v^x + (ul)_{(x)+1}v^{x+1} + (ul)_{(x)+2}v^{x+2} + (ul)_{(x)+3}v^{x+3} + (ul)_{(x)+4}v^{x+4}\} \div (ul)_x v^x;$$

and proceeding as before, it is easy to see that we have

$$(u\mathbf{a})_x = \frac{(u\mathrm{N})_x}{(u\mathrm{D})_x} = \frac{\mathrm{N}_x - \mathrm{N}_{[x]}}{\mathrm{D}_x - \mathrm{D}_{[x]}}.$$

Since $(u\mathbf{a})_x$ and $(u\mathrm{A})_x$ are the ordinary whole-life annuity-due and assurance, on lives which are subject to a certain rate of mortality, the usual well-known relation will exist between them, $\mathrm{A} = 1 - 1 - d\mathbf{a}$.

The following table gives specimen values of these functions.

x (1)	$(u\mathrm{D})_x$ (2)	$(u\mathrm{N})_x$ (3)	$(u\mathrm{M})_x$ (4)	$(u\mathbf{a})_x$ (5)	$(u\mathrm{A})_x$ (6)	x (7)
30	1997·91	2865·78	1887·68	1·4344	·94483	30
35	2248·30	4158·85	2088·35	1·8498	·92886	35
40	1771·50	2943·86	1658·27	1·6618	·93608	40
45	1884·55	3657·68	1743·87	1·9409	·92535	45
50	2037·10	3891·82	1887·41	1·9105	·92652	50
55	1756·46	3057·70	1638·86	1·7408	·93305	55
60	1581·52	2494·91	1485·56	1·5775	·93933	60
65	1347·71	1784·58	1279·07	1·3242	·94907	65
70	1149·66	1491·07	1092·31	1·2970	·95012	70
75	1131·05	1474·24	1074·35	1·3034	·94987	75

We have seen that the l_x lives of the $\mathrm{H}^{\mathrm{M}(5)}$ Table include $(ul)_x$ damaged lives and $l_{[x]}$ lives which are still select. It will be convenient in what follows to use the symbol $(hl)_x$ sometimes instead of $l_{[x]}$, altering similarly the expressions for the annuities, the assurances, and the commutation functions. It is clear that the value of the annuities upon the l_x mixed lives of any age must be equal to the sum of the annuities on the corresponding $(hl)_x$ select lives and $(ul)_x$ damaged lives. The same thing appears from the formulas that we have obtained.

We have

$$(hl)_x.(h\mathbf{a})_x = (hl)_x.\frac{(hN)_x}{(hD)_x} = (hl)_x.\frac{(hN)_x}{(hl)_x v^x} = \frac{1}{v^x}(hN)_x.$$

Also $(ul)_x.(u\mathbf{a})_x = (ul)_x.\dfrac{(uN)_x}{(uD)_x} = (ul)_x.\dfrac{(uN)_x}{(ul)_x v^x} = \dfrac{1}{v^x}(uN)_x.$

$$\therefore (hl)_x.(h\mathbf{a})_x + (ul)_x.(u\mathbf{a})_x = \frac{1}{v^x}(hN)_x + (uN)_x\} = \frac{1}{v^x}N_x = l_x.\mathbf{a}_x.$$

In a similar way we may prove that $(hl)_x.(hA)_x + (ul)_x.(uA)_x = l_x.A_x.$

We have a corresponding formula for the policy-values. Let P be the premium valued in finding the policy-value, and let ${}_n(uV)_x$ denote the value of a policy taken out n years ago (n being >4), on a life which was then of the age x, and is now one of the damaged lives of our table and of the age $x+n$. Then ${}_n(uV)_x = (uA)_{x+n} - (u\mathbf{a})_{x+n}.P$; also ${}_n(hV)_x = (hA)_{x+n} - (h\mathbf{a})_{x+n}.P$, and ${}_nV_x = A_{x+n} - \mathbf{a}_{x+n}.P$; then by means of the relations proved above,

$$(hl)_{x+n}.{}_n(hV)_x + (ul)_{x+n}.{}_n(uV)_x = (hl)_{x+n}\{(hA)_{x+n} - (h\mathbf{a})_{x+n}.P\}$$
$$+ (ul)_{x+n}\{(uA)_{x+n} - (u\mathbf{a})_{x+n}.P\}$$
$$= (hl)_{x+n}(hA)_{x+n} + (ul)_{x+n}(uA)_{x+n} - \{(hl)_{x+n}(h\mathbf{a})_{x+n} + (ul)_{x+n}(u\mathbf{a})_{x+n}\}\ P$$
$$= l_{x+n}A_{x+n} - l_{x+n}\mathbf{a}_{x+n}P = l_{x+n}.{}_nV_x.$$

Hence we get ${}_n(uV)_x = \dfrac{l_{x+n}.{}_nV_x - l_{[x+n]}.{}_n(hV)_x}{l_{x+n} - l_{[x+n]}}.$

We see that these relations hold whatever net premium is valued.

A numerical example will perhaps place them in a clearer light. If the premium valued is $\pi_{[x]}$, then 100 ${}_nV_x$ will be the average policy value tabulated in col. (4) of the table on p. 23, and 100 ${}_n(hV)_x$ will be the value tabulated in col. (5) of the same table. As an illustration of the above theory, we will take the case of a policy effected at the age of 30 which has been 15 years in force. We have $\pi_{[30]} = \cdot01714$ (p. 21); $(hl)_{45} = 730692$,

$(ul)_{45} = 11008$, $l_{45} = 741700$ (p. 52); ${}_{15}V_{[30]} = \cdot 18048$, ${}_{15}(hV)_{30} = \cdot 16976$ (p. 23); $(u\mathbf{a})_{45} = 1\cdot 9409$, $(uA)_{45} = \cdot 92535$ (p. 42); whence ${}_{15}(uV)_{30} = (uA)_{45} - (u\mathbf{a})_{45}\ \pi_{[30]} = \cdot 92535 - \cdot 03327 = \cdot 89208$.

Hence

$$\begin{array}{rcrcrcr}
(hl)_{45} \times {}_{15}(hV)_{30} & = & 730692 & \times & \cdot 16976 & = & 124042 \\
(ul)_{45} \times {}_{15}(uV)_{30} & = & 11008 & \times & \cdot 89208 & = & 9820 \\
\hline
l_{45} \times {}_{15}V_{30} & = & 741700 & \times & \cdot 18048 & = & 133862
\end{array}$$

We have now to consider the $(ul)_{[x]+n}$ damaged lives contained in $l_{[x]+n}$ mixed lives, where $n < 5$. If we extend the notation of the commutation columns in the same way as we have extended the notation for the numbers living and dying, we have

$$(uD)_{([x]+n)+t} = v^{x+n+t}.(ul)_{([x]+n+t)} = v^{x+n+t}(l_{[x]+n+t} - l_{[x+n]+t})$$
$$= D_{[x]+n+t} - D_{[x+n]+t}.$$

But it is only necessary to deal with the case of $t=0$, when we have $(uD)_{[x]+n} = D_{[x]+n} - D_{[x+n]}$.

By similar reasoning, $(u\mathrm{N})_{[x]+n} = \mathrm{N}_{[x]+n} - \mathrm{N}_{[x+n]}$,

and $$(uM)_{[x]+n} = M_{[x]+n} - M_{[x+n]}.$$

Furthermore, $(u\mathbf{a})_{[x]+n} = \dfrac{(u\mathrm{N})_{[x]+n}}{(uD)_{[x]+n}} = \dfrac{\mathrm{N}_{[x]+n} - \mathrm{N}_{[x+n]}}{D_{[x]+n} - D_{[x+n]}}$;

and $$(uA)_{[x]+n} = \frac{(uM)_{[x]+n}}{(uD)_{[x]+n}} = \frac{M_{[x]+n} - M_{[x+n]}}{D_{[x]+n} - D_{[x+n]}}.$$

It is also easy to see that we have

$$(uM)_{[x]+n} = (uD)_{[x]+n} - d(u\mathrm{N})_{[x]+n},$$

and $$(uA)_{[x]+n} = 1 - d(u\mathbf{a})_{[x]+n}.$$

Also

$$(hl)_{[x]+n}.(h\mathbf{a})_{[x]+n}=(hl)_{[x]+n}.\frac{(h\mathrm{N})_{[x]+n}}{(h\mathrm{D})_{[x]+n}}=\frac{(hl)_{[x]+n}.(h\mathrm{N})_{[x]+n}}{(hl)_{[x]+n}.v^{x+n}}=\frac{1}{v^{x+n}}.(h\mathrm{N})_{[x]+n},$$

$$(ul)_{[x]+n}.(u\mathbf{a})_{[x]+n}=(ul)_{[x]+n}.\frac{(u\mathrm{N})_{[x]+n}}{(u\mathrm{D})_{[x]+n}}=\frac{(ul)_{[x]+n}.(u\mathrm{N})_{[x]+n}}{(ul)_{[x]+n}.v^{x+n}}=\frac{1}{v^{x+n}}.(u\mathrm{N})_{[x]+n};$$

$$\therefore\quad (hl)_{[x]+n}.(h\mathbf{a})_{[x]+n}+(ul)_{[x]+n}.(u\mathbf{a})_{[x]+n}=\frac{1}{v^{x+n}}\left\{(h\mathrm{N})_{[x]+n}+(u\mathrm{N})_{[x]+n}\right\}$$

$$=\frac{1}{v^{x+n}}.\mathrm{N}_{[x]+n}=l_{[x]+n}.\mathbf{a}_{[x]+n}.$$

Similarly we may show that $(hl)_{[x]+n}.(h\mathrm{A})_{[x]+n}+(ul)_{[x]+n}.(u\mathrm{A})_{[x]+n}.$ $=l_{[x]+n}.\mathrm{A}_{[x]+n}$. From these relations it at once follows that, whatever the net premium valued, $(hl)_{[x]+n}.{}_n(h\mathrm{V})_x+(ul)_{[x]+n}.{}_n(u\mathrm{V})_x$ $=l_{[x]+n}.{}_n\mathrm{V}_x$. When $n=$ or >5, all these formulas agree with those previously obtained.

As another instance of the use of our tables in calculating the value of options to be exercised by the assured, let us suppose that, as part of some business arrangement, an office enters into an agreement that it will upon application n years hence, if a person now of the age x shall be then alive, grant a policy on his life at the ordinary rate of premium. If the person is alive and still in good health, it is clear the company will be at no disadvantage in then granting a policy at the ordinary rate of premium. The risk, therefore, which the company runs is, that the life will be still in existence at the end of n years, but have become damaged; and we have seen that the probability of this is

$$\frac{(ul)_{x+n}}{l_{[x]}}=\frac{l_{x+n}-l_{[x+n]}}{l_{[x]}}$$

Instead of confining ourselves to the case of a single contract of this kind, it is convenient, as in most questions of life contingencies, to suppose that the company enters into $l_{[x]}$ contracts

of the kind. Then our table shows us that at the end of n years there will be $(ul)_{x+n}$ damaged lives who will die within 5 years; and the sum which the company should have in hand at the outset in order to provide for these claims is $(ud)_{x+n}v^{n+1} + (ud)_{(x+n)+1}v^{n+2} + (ud)_{(x+n)+2}v^{n+3} + (ud)_{(x+n)+3}v^{n+4} + (ud)_{(x+n)+4}v^{n+5}$, and the single premium therefore to be charged in the case of one life is the same quantity divided by $l_{[x]}$

$$\frac{1}{l_{[x]}v^x}\Big((ud)_{x+n}v^{x+n+1} + (ud)_{(x+n)+1}v^{x+n+2} + (ud)_{(x+n)+2}v^{x+n+3} + (ud)_{(x+n)+3}v^{x+n+4} + (ud)_{(x+n)+4}v^{x+n+5}\Big) = \frac{(uM)_{x+n}}{D_{[x]}} = \frac{(uD)_{x+n}}{D_{[x]}}\cdot\frac{(uM)_{x+n}}{(uD)_{x+n}}$$

$= \frac{(uD)_{x+n}}{D_{[x]}}.(uA)_{x+n}$. The analogy of this formula to an ordinary deferred assurance is obvious. Hitherto we have left out of account the premiums which the company is to receive. Suppose for the present that this premium is P. Then the value of the premiums to be received on the damaged lives will be

$$\{(ul)_{x+n}v^n + (ul)_{(x+n)+1}v^{n+1} + (ul)_{(x+n)+2}v^{n+3} + (ul)_{(x+n)+3}v^{n+4} + (ul)_{(x+n)+4}v^{n+5}\}\,P,$$

and the chance of the company receiving such premiums in respect of a particular nominated life is this quantity divided by $l_{[x]}$

$$\frac{(ul)_{x+n}v^{x+n} + (ul)_{(x+n)+1}v^{x+n+1} + (ul)_{(x+n)+2}v^{x+n+2} + (ul)_{(x+n)+3}v^{x+n+3} + (ul)_{(x+n)+4}v^{x+n+4}}{l_{[x]}v^x}$$

$$= \frac{(uN)_{x+n}}{D_{[x]}}.\,P = \frac{(uD)_{x+n}}{D_{[x]}}\cdot\frac{(uN)_{x+n}}{uD_{x+n}}.\,P = \frac{(uD)_{x+n}}{D_{[x]}}(u\mathrm{a})_{x+n}.P,$$

being analogous to an ordinary deferred annuity. We may use the symbols ${}_n(u\mathrm{a})_{[x]}$ and ${}_n(uA)_{[x]}$ to denote the deferred annuities and assurances we have been here considering. Combining our results, the single premium that the company should receive at the outset in consideration of entering into the supposed contract is

$$\frac{(u\mathrm{D})_{x+n}}{\mathrm{D}_{[x]}}\cdot\Big((u\mathrm{A})_{x+n}-(u\mathbf{a})_{x+n}.\mathrm{P}\Big)\text{ and if } \mathrm{P}=\pi_{[x+n]},\text{ then}$$

$$(u\mathrm{A})_{x+n}-(u\mathbf{a})_{x+n}.\mathrm{P}=(u\mathrm{A})_{x+n}-(u\mathbf{a})_{x+n}.\pi_{[x+n]}=(u\mathbf{a})_{x+n}\{(u\pi)_{x+n}-\pi_{[x+n]}\}$$
$$=1-\frac{(u\mathbf{a})_{x+n}}{\mathbf{a}_{[x+n]}}.$$

On the supposition that the risk has turned out in favour of the assured and against the company, this quantity is the value to the assured of the policy on a damaged life at the date of its issue, and may be denoted by the symbol $(u\mathrm{V})_{(x+n)}$. Then the single premium that the company should receive at the outset is $\frac{(u\mathrm{D})_{x+n}}{\mathrm{D}_{[x]}}.(u\mathrm{V})_{(x+n)}$. The value of an annuity-due for n years upon $[x]$ being $\frac{\mathrm{N}_{[x]}-\mathrm{N}_{x+n}}{\mathrm{D}_{[x]}}$, it follows that the annual premium payable for n years which should be charged by the company as the consideration for the risk is $\frac{(u\mathrm{D})_{x+n}}{\mathrm{N}_{[x]}-\mathrm{N}_{x+n}}.(u\mathrm{V})_{(x+n)}$. It must be borne in mind, however, that, if the company agree to accept this annual premium in consideration of undertaking the risk supposed, and the assured retain the usual right of dropping the policy, he will be able to exercise an option which has not entered into the calculation; and the company, therefore could not agree to accept this annual premium, unless it were in some way protected against the risk of the option being exercised to its disadvantage. Such protection would be obtained by making the above annual premium an addition to the ordinary premium for an ordinary policy.

As a numerical example of the above theory, let it be required to find the net single premium that a company should receive at the outset, in consideration of entering into a contract that it will, upon application 20 years hence, if a person now of the age of 30 shall be then alive, grant a policy on his life at the

ordinary rate of premium. The single premium will be

$$\frac{(uD)_{50}}{D_{[30]}}.(uV)_{(50)}=\frac{(uD)_{50}}{D_{[30]}}\left(1-\frac{(u\mathbf{a})_{50}}{\mathbf{a}_{[50]}}\right)=\frac{2037\cdot1}{266022}\times\left(1-\frac{1\cdot9105}{13\cdot6346}\right)$$

$= \cdot0076576 \times \cdot85988 = \cdot0065846 =$ about 13s. 2d. per cent.

The annual premium will be about 1s. per cent.

As a final illustration of the method of calculating the value of options that may be exercised by the assured, I will now find the formula for a short term assurance, on the supposition that all the healthy lives drop their policies at the end of a year, leaving the office to pay the claims upon all the damaged lives. Suppose that $l_{[x]}$ persons effect insurances on their lives, and, first, that the term of the insurances is six or more years, the effect of which latter supposition is, that all the damaged lives at the end of the first year will die during the term for which the insurance is current. Then the number of deaths in the first year will be $d_{[x]}$, and the number of survivors at the end of the year will be $l_{[x]+1}$, of whom $l_{[x+1]}$ are still in good health and, according to our supposition, drop their policies, leaving on the company's books the damaged lives, in number, $(ul)_{[x]+1}=l_{[x]+1}-l_{[x+1]}$. Then we see that the sum which would at the outset be sufficient tô provide for payment of the claims is

$$d_{[x]}.v+(ud)_{[x]+1}v^2+(ud)_{([x]+1)+1}v^3+(ud)_{([x]+1)+2}v^4+(ud)_{([x]+1)+3}v^5 \\ +(ud)_{([x]+1)+4}v^6$$

$$=d_{[x]}v+(d_{[x]+1}-d_{[x+1]})v^2+(d_{[x]+2}-d_{[x+1]+1})v^3+(d_{[x]+3}-d_{[x+1]+2})v^4 \\ +(d_{[x]+4}-d_{[x+1]+3})v^5+(d_{[x]+5}-d_{[x+1]+4})v^6=(M_{[x]}-M_{[x+1]})\div v^x.$$

By a similar process of reasoning it can be proved that, if P is the premium paid by each of the $l_{[x]}$ lives, then the value of all the premiums receivable on the suppositions made, is $(N_{[x]}-N_{[x+1]})P\div v^x$; and equating the value of the premiums to the value of the claims, we get $P=\frac{M_{[x]}-M_{[x+1]}}{N_{[x]}-N_{[x+1]}}$.

Next let us suppose that all the healthy lives drop their policies at the end of the second year, the original term of the insurance being 7 or more years, so that all the damaged lives will die within the term of the insurance. Then it may be proved by a process of reasoning strictly analogous to the foregoing that the annual premium to provide for the claims is $\dfrac{M_{[x]}-M_{[x+2]}}{N_{[x]}-N_{[x+2]}}$; and in general, if the healthy lives drop their policies at the end of the tth year, the original term of the insurance being $t+5$ years or more, then the annual premium that will provide for the claims is $\dfrac{M_{[x]}-M_{[x+t]}}{N_{[x]}-N_{[x+t]}}$. The form of this expression shows us that the numerator corresponds to the claims among the $l_{[x]}$ lives under deduction of the claims among the $l_{[x+t]}$ lives who are still select after the lapse of t years; that is to say, to all the claims during the t years and to the claims among the damaged lives thereafter, and the denominator corresponds to the premiums payable in respect of the same lives.

The formula will take a different shape when n, the original term of the insurance, $< t+5$. Suppose, for instance, that the original term of the insurance is n years, and that the healthy lives withdraw at the end of the tth year; then it is easy to see that the sum which is necessary to provide for the claims during the first t years is $(M_{[x]}-M_{[x]+t})\div v^{x}$, and the sum necessary to provide for the claims arising from the deaths of damaged lives in the following $n-t$ years is

$$(M_{[x]+t}-M_{[x+t]}-M_{[x]+n}+M_{[x+t]+n-t})\div v^{x}.$$

Adding these together the total sum necessary to provide for the claims is

$$(M_{[x]}-M_{[x+t]}-M_{[x]+n}+M_{[x+t]+n-t})\div v^{x}.$$

Similarly, if P is the premium payable in respect of each

insurance, the value of all the premiums at the outset will be

$$(\mathrm{N}_{[x]}-\mathrm{N}_{[x+t]}-\mathrm{N}_{[x]+n}+\mathrm{N}_{[x+t]+n-t})\mathrm{P}\div v^{x};$$

and equating these quantities, we get

$$\mathrm{P}=\frac{\mathrm{M}_{[x]}-\mathrm{M}_{[x+t]}-\mathrm{M}_{[x]+n}+\mathrm{M}_{[x+t]+n-t}}{\mathrm{N}_{[x]}-\mathrm{N}_{[x+t]}-\mathrm{N}_{[x]+n}+\mathrm{N}_{[x+t]+n-t}}.$$

Here the terms of the numerator, $\mathrm{M}_{[x]}-\mathrm{M}_{[x]+n}$, correspond to the whole of the claims that arise out of $l_{[x]}$ persons in n years, and the other terms, $\mathrm{M}_{[x+t]}-\mathrm{M}_{[x+t]+n-t}$, correspond to the claims during the last $(n-t)$ years out of the lives which are still select at the end of t years; and the difference between these is clearly the whole of the claims for t years and the claims in respect of the damaged lives during the other $n-t$ years. Similarly for the denominator.

Next, let us suppose that policies are granted on the half-premium system, so that at the end of five years the premium is doubled, remaining thereafter constant; and let us further suppose that at the end of five years all the healthy lives withdraw. Then reasoning as before, the number of lives assured being $l_{[x]}$, the sum necessary to provide for the claims during the five years is $(\mathrm{M}_{[x]}-\mathrm{M}_{x+5})\div v^{x}$; and the sum necessary to provide for the claims among the damaged lives, after the expiry of five years, is $(\mathrm{M}_{x+5}-\mathrm{M}_{[x+5]})\div v^{x}$. Therefore the total sum necessary to provide for the claims is $(\mathrm{M}_{[x]}-\mathrm{M}_{[x+5]})\div v^{x}$. If P is the premium payable during the first five years, which is increased to 2P subsequently, the value of the premiums receivable will be

$$\frac{\mathrm{N}_{[x]}-\mathrm{N}_{x+5}}{v^{x}}.\mathrm{P}+\frac{\mathrm{N}_{x+5}-\mathrm{N}_{[x+5]}}{v^{x}}.2\mathrm{P}=\frac{\mathrm{N}_{[x]}+\mathrm{N}_{x+5}-2\mathrm{N}_{[x+5]}}{v^{x}}.\mathrm{P}.$$

Equating the two quantities, we get $\mathrm{P}=\dfrac{\mathrm{M}_{[x]}-\mathrm{M}_{[x+5]}}{\mathrm{N}_{[x]}+\mathrm{N}_{x+5}-2\mathrm{N}_{[x+5]}}$.

This quantity is the premium that will exactly suffice to meet the claims according to the suppositions we have made; but it does not follow that it will be the proper premium to charge. In order to decide as to this, we must calculate the increasing premium in the ordinary way by the formula $P' = \frac{M_{[x]}}{N_{[x]} + N_{x+5}}$.

Then, if $P < P'$, we learn that we may safely charge the premium P'; but if $P > P'$, the latter premium will not be sufficient to meet the risk during the first five years. We might then charge the premium P during the five years; but it would be unnecessary to charge 2 P for the remainder of life, and the proper course would be to charge for the remainder of life the premium P'' determined by the equation

$$P\,(N_{[x]} - N_{x+5}) + P''.N_{x+5} = M_{[x]}$$

Extract from Table No. 1, being an extended Table of the Values of l_x for different Ages at Entry and different Periods since Entry; showing also the number of Damaged Lives contained in them.

Age.	Select Lives.	Mixed Lives.					Damaged Lives.										Age.
	Numbers Living.						Numbers Living.					Numbers Dying.					
	Years elapsed since Date of Insurance.																
	0	1	2	3	4	5 or more											
x	$l_{[x]}$	$l_{[x-1]+1}$	$l_{[x-2]+2}$	$l_{[x-3]+3}$	$l_{[x-4]+4}$	l_x	$l_x - l_{[x]} = (ul)_x$	$l_x - l_{[x-1]+1} = (ul)_{(x-1)+1}$	$l_x - l_{[x-2]+2} = (ul)_{(x-2)+2}$	$l_x - l_{[x-3]+3} = (ul)_{(x-3)+3}$	$l_x - l_{[x-4]+4} = (ul)_{(x-4)+4}$	$d_x - d_{[x]} = (ud)_x$	$d_x - d_{[x-1]+1} = (ud)_{(x-1)+1}$	$d_x - d_{[x-2]+2} = (ud)_{(x-2)+2}$	$d_x - d_{[x-3]+3} = (ud)_{(x-3)+3}$	$d_x = d_{[x-4]+4} = (ud)_{(x-4)+4}$	x
30	862820	866862	868784	869300	869300	869300	6480	2438	516	..	..	4116	1838	506	..	..	30
31	854970	858936	860700	861290	861300	861300	6330	2364	600	10	..	4006	1778	516	10	..	31
32	846500	851076	852814	853316	853400	853400	6900	2324	586	84	..	3986	1794	586	84	..	32
33	838098	842586	844970	845500	845500	845500	7402	2914	530	..	..	3856	1692	530	..	..	33
34	829492	834154	836478	837700	837700	837700	8208	3546	1222	..	..	3918	1772	644	..	..	34
35	820928	825510	828026	829222	829800	829800	8872	4290	1774	578	..	4278	2142	1030	416	..	35
36	812338	816906	819352	820756	821338	821500	9162	4594	2148	744	162	4435	2304	1198	512	162	36
37	803767	808273	810710	812050	812768	813000	9233	4727	2290	950	232	4589	2457	1358	610	232	37
38	795196	799656	802030	803368	803960	804300	9104	4644	2270	932	340	4738	2606	1504	726	340	38
39	786608	791034	793362	794634	795194	795400	8792	4366	2038	766	206	4682	2543	1438	654	206	39
40	777995	782390	784677	785900	786388	786500	8505	4110	1823	600	112	4622	2472	1357	600	112	40
41	769147	773717	775962	777134	777600	777600	8453	3883	1638	466	..	4455	2289	1163	444	..	41
42	760067	764802	767206	768325	768778	768800	8733	3998	1594	475	22	4483	2296	1152	453	22	42
43	750725	755650	758198	759458	759878	759900	9175	4250	1702	442	22	4505	2292	1126	409	22	43
44	740943	746230	748942	750324	750867	750900	9957	4670	1958	576	33	4620	2373	1183	414	33	44
45	730692	736363	739403	740925	741538	741700	11008	5337	2297	775	162	4923	2639	1417	570	162	45
46	720100	726015	729402	731220	731895	732100	12000	6085	2698	880	205	5220	2891	1631	688	205	46
47	709190	715320	718906	721033	721908	722100	12910	6780	3194	1067	192	5502	3122	1819	790	192	47
48	698088	704292	708042	710325	711423	711700	13612	7408	3658	1375	277	5870	3431	2078	988	277	48
49	686620	693058	696823	699220	700413	700800	14180	7742	3977	1580	387	6225	3719	2305	1172	387	49
50	674923	681445	685377	687728	688992	689400	14477	7955	4023	1672	408	6467	3879	2393	1238	408	50
51	662891	669590	673524	675970	677166	677600	14709	8010	4076	1630	434	6695	4011	2449	1278	434	51
52	650511	657386	661401	663773	665048	665400	14889	8014	3999	1627	352	6806	4016	2366	1184	352	52
53	637720	644817	648902	651267	652457	652900	15180	8083	3998	1633	443	7101	4192	2439	1247	443	53
54	624683	631821	636009	638341	639514	639900	15217	8079	3891	1559	386	7280	4239	2370	1169	386	54
55	611313	618563	622660	624979	626110	626500	15187	7937	3840	1521	390	7543	4353	2354	1155	390	55
56	597374	604956	609016	611114	612234	612600	15226	7644	3584	1486	366	7792	4434	2285	1095	366	56
57	582899	590766	594990	596901	597809	598200	15301	7434	3210	1299	391	8128	4587	2268	1057	391	57
58	567710	[illegible]76027	580353	582258	582958	583200	15490	7173	2847	942	242	8356	4613	2104	830	242	58
59	551749	5[illegible]566	565140	566957	567588	567700	15951	7134	2560	743	112	8677	4715	2003	633	112	59
60	534963	544[illegible]26	549181	551043	551490	551600	16637	7274	2419	557	110	9196	4996	2057	557	110	60

N.B.—The figures in larger type are those of the $H^{M(5)}$ Table. [Eds.]